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Algebraic Description of the Scattering of a Spin 1/2 Particle off a Dyon

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Abstract

A spin 1/2 particle scattered off a dyon is described by Zwanziger's algebraic approach, based on the $o(3,1)\oplus o(3)$dynamical symmetry discovered by D'Hoker and Vinet. The S-matrix is shown to factorize into the product of the spinless S-matrix S_0 with a spin-dependent factor and the total cross-section is identical to the one found in the spinless case.

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The field of a Dirac monopole does *not* have a time-independent o(3,1) dynamical symmetry analogous to that of the Kepler problem[1]. However, as first pointed out by Zwanziger[2] twenty years ago, a spinless particle moving in the combined field of an electrically charged Dirac monopole and of a appropriate $1/r^2$ potential (called a *dyon* in what follows) *does* have such a symmetry[2,3,4]. The particle is described by the Hamiltonian

$$H_o = (1/2)\,(p_i - eA_i)^2 - 1/r + q^2/2r^2 + 1/2q^2, \tag{1}$$

where $p_k = -i\partial_k$, $(\text{rot}\,\mathbf{A})^i = B_i = g r^i/r^3$ and $q = eg$ is a half-integer. At first sight this system seems to be rather artificial. However, the equations of motion of a spinless test particle in the asymptotic field of a Bogomolny-Prasad-Sommerfield monopole reduce exactly to these equations[4].

The monopole and $1/r^2$ terms in (1) combine in a particular way[2,3,4] and the system admits indeed, in addition to angular momentum,

$$L^i = \varepsilon^{ijk} r^j v^k - q\, r^i/r\,, \qquad (\text{ where } \mathbf{v} = \mathbf{p} - e\mathbf{A}\,) \tag{2}$$

also a conserved 'Runge-Lenz' vector,

$$\Lambda_o{}^i = (1/2)\,\varepsilon^{ijk}\,(v^j L^k - L^j v^k) - r^i/r\,. \tag{3}$$

For fixed value E of the energy, the rescaled Runge-Lenz vector $\mathbf{K} = (1-2q^2E)^{-1/2}\, q\Lambda_o$ extends the manifest o(3) symmetry into a dynamical o(4) algebra. The well-known Pauli-Bargmann[5] method (originally introduced for the Hydrogen atom) yields[2,3,4] the (p^2-q^2)-fold degenerate bound-state energy spectrum

$$E = 1/2q^2 - 1/2p^2, \qquad p = |q|+1,\ |q|+2,\ldots \tag{4}$$

For energies larger than $1/2q^2$, $\mathbf{J}$ and $\mathbf{K} = (2q^2E-1)^{-1/2}\, q\Lambda_o$ close rather to o(3,1), and the less-well-known method of Zwanziger[2] allows for an algebraic derivation of the S-matrix. The result only depends on the Casimirs

$$\mathbf{J}.\mathbf{K} = |q|\,/\,k\,, \qquad \mathbf{J}^2 - \mathbf{K}^2 = q^2 - 1/k^2 - 1, \tag{5}$$

and is found to be

$$S_o(\mathbf{l},\mathbf{k}) = \langle \mathbf{l}(\text{out})|\, \mathbf{k}(\text{in})\rangle$$

$$= \delta(E_l - E_k) \sum_{j \geq |q|} (2j+1)\left[\frac{(j-i/k)!}{(j+i/k)!}\right] D^j_{(-q,q)}\,(R^{-1}{}_l\, R_{\mathbf{k}}), \tag{6}$$

where $\mathbf{k}$ is the wave vector and E_k is the corresponding energy,

$$k = |\mathbf{k}| = (2E - 1/q^2)^{1/2} \quad \text{i. e.} \quad E_k = k^2/2 + 1/2q^2, \tag{7}$$

and $R_{\mathbf{k}}$ is a rotation matrix which brings the z-axis into the direction of $\mathbf{k}$. Eqn. (6) leads to the modified Rutherford cross-section

$$d\sigma_o/d\omega = \{\,(1+ (qk)^2)/4k^4\,\}\ \text{cossec}^4\,\theta/2\,. \tag{8}$$

The bound-state spectrum can be re-derived from the poles of S-matrix: the condition $j\text{-}ia = -n$ yields (using the notation $j+n = p$), eqn. (4) once more. (The method also works for monopole scattering[6]).

Recently D'Hoker and Vinet[7] have suggested to consider a *spin-1/2* particle with gyromagnetic ratio 4, with Pauli Hamiltonian

$$H = H_o - e\,B^i\sigma^i. \tag{9}$$

The system (9) also arises in the study of a spin 1/2 particle in the field of a Bogomolny-Prasad-Sommerfield monopole[8].

For energy $E < 1/q^2$ the system has again bound states. Remarkably, its positive-energy spectrum is *identical* to that in (4), with $2(p^2-q^2)$-fold degeneracy, twice that in the spinless case. A novel feature is that it also has E=0 ground-states whose degeneracy is $2|q|$.

This remarkable result is explained by *supersymmetry*: D'Hoker and Vinet construct in fact[7] a supermultiplet consisting of two spinless and a spin 1/2 particles, whose dynamics are governed by H_o and by H, respectively. The system admits a large dynamical supersymmetry algebra. Both the identity of the positive spectra and the arising of a symmetry-breaking E = O ground state are typical manifestations of the spectral (a)symmetry of supersymmetric quantum mechanics: $2|q|$ is just the Atiyah-Singer (alias Witten) index.

In the Kepler case the introduction of spin destroys the dynamical symmetry. For the Zwanziger system it just the contrary of this what happens. Indeed, D'Hoker and Vinet show in another paper[9] that the system (9) is even *more symmetric* and it admits rather *three* conserved vectors,

namely

$$J^i = L^i + \sigma^i/2 \; ; \tag{10a}$$

$$\Lambda^i = \Lambda_o{}^i - \varepsilon^{ijk}\, \sigma^j v^k + (q/r)\, \sigma^i - q\, r^i (r^j/r^3)\, \sigma^j - (1/2q)\, \sigma^i \; ; \tag{10b}$$

$$\Omega^i = (1/2)\, \sigma^i v^2 - \sigma^j v^j v^i + \varepsilon^{ijk} (q/r - 1/q)\, \sigma^j v^k - (1/2)(q/r - 1/q)^2 \sigma^i \, . \tag{10c}$$

(This algebra is in fact inheritied from the bosonic part of the D'Hoker-Vinet dynamical superalgebra in Ref. 7.).

Consider rather the combinations

$$J_1{}^i = J^i + (1/2H)\, \Omega^i \tag{11a}$$

$$K^i = (1-2q^2H)^{-1/2} (q\Lambda^i - (1/2H)\, \Omega^i) \tag{11b}$$

$$J_2{}^i = -(1/2H)\, \Omega^i . \tag{11c}$$

For a fixed value $0 < E < 1/2q^2$ of the energy, the operators (11) satisfy the commutation relations

$$\begin{aligned} &[J_1{}^i, J_1{}^j] = i\, \varepsilon^{ijk} J_1{}^k, \quad [J_1{}^i, K^j] = i\, \varepsilon^{ijk} K^k, \quad [K^i, K^j] = i\, \varepsilon^{ijk} J_1{}^k \\ \text{and} \quad &[J_2{}^i, J_2{}^j] = i\, \varepsilon^{ijk} J_2{}^k , \quad [J_2{}^i, J_1{}^j] = [J_2{}^i, K^j] = 0 , \end{aligned} \tag{12}$$

so they span an $o(4)\oplus o(3)$ dynamical symmetry algebra. Observe that the extra $o(3)$ comes entirely from the spin. The Pauli-Bargmann method[5] allows then for another derivation of the bound state spectrum[8]. Notice that the extra factor 2 in the degeneracy arises due to the extra $o(3)$ in its spin 1/2 representation. (For the ground-state $E=0$ the symmetry is $E(3)\oplus o(3)$, and the degeneracy is entirely due to angular momentum, $2j+1 = 2(|q|-1/2)+1 = 2|q|$, see Ref. 9).

This Letter is devoted to an *algebraic description* of the *scattering* of a spin 1/2 particle by a dyon, based on its *dynamical symmetry*. The main result is that the S matrix *factorizes* as a product of the spinless S-matrix S_o with a factor which only depends on the spin degrees of freedom. The poles of S and S_o coincide up to multiplicity (as they should), yielding identical positive spectra.

Indeed, let us restrict ourselves in the sequel to those states of fixed

$E > 1/2q^2$ value of the energy. The definition of K^i is now rather

$$K^i = (2q^2E-1)^{-1/2} (q\Lambda^i - (1/2E)\, \Omega^i) \tag{11b'}$$

The commutation relations become then

$$\begin{aligned} &[J_1{}^i, J_1{}^j] = i\, \varepsilon^{ijk} J_1{}^k, \quad [J_1{}^i, K^j] = i\, \varepsilon^{ijk} K^k, \quad [K^i, K^j] = -\, i\, \varepsilon^{ijk} J_1{}^k \\ \text{and} \quad &[J_2{}^i, J_2{}^j] = i\, \varepsilon^{ijk} J_2{}^k , \quad [J_2{}^i, J_1{}^j] = [J_2{}^i, K^j] = 0, \end{aligned} \tag{12'}$$

showing that $\mathbf{J}_1$ and $\mathbf{K}$ span now an $o(3,1)$ algebra (generalizing that of the spinless case) to which $\mathbf{J}_2$ adds an independent $o(3)$. So the dynamical symmetry is now rather $o(3,1)\oplus o(3)$. The extra $o(3)$ will be used for fixing the helicity, needed for the Zwanziger-type calculation of the S-matrix, based on $o(3,1)$. The Casimir operators are now calculated[9] to be

$$\mathbf{J}_1.\mathbf{K} = |q|/k , \qquad (\mathbf{J}_1)^2 - \mathbf{K}^2 = q^2 - 1/k^2 - 1 , \qquad (\mathbf{J}_2)^2 = 3/4 , \tag{13}$$

with k still given formally by eqn. (7). Remarkably, the Casimirs of the $o(3,1)$ in particular take the same values as for spin 0, cf. eqn. (5).

The Casimir operators single out the actual representation of $o(3,1)\oplus o(3)$ which operates on the scattering states. The symmetry generators allow for a complete labeling of the states, so we have an irreducible representation (spanned, of course, by non-renormalizable basis vectors). The $o(3)$ of $\mathbf{J}_2$ is in its spin-1/2 representation. Representation theory[10] tells us then that among the scattering states (solutions of the time-dependent Schrödinger equation) we have a convenient 'distorted spherical-wave' basis $| k, j_1, m, \mu >$, defined by the relations

$$H \,| k, j_1, m, \mu > \; = E_k \,| k, j_1, m, \mu > , \qquad k > 0, \tag{14a}$$

$$\mathbf{J}_1.\mathbf{J}_1 \,| k, j_1, m, \mu > = j_1(j_1+1) \,| k, j_1, m, \mu > , \qquad j_1 = |q|, |q|+1, \ldots ; \tag{14b}$$

$$J_1{}^3 \,| k, j_1, m, \mu > \; = m \,| k, j_1, m, \mu > , \qquad m = -j_1, \ldots, j_1 ; \tag{14c}$$

$$J_2{}^3 \,| k, j_1, m, \mu > \; = \mu \,| k, j_1, m, \mu > , \qquad \mu = \pm 1/2 \; ; \tag{14d}$$

$$< k', j_1', m', \mu' \,| k, j_1, m, \mu > \; = \delta(E_{k'} - E_k)\, \delta_{j'j}\, \delta_{m',m}\, \delta_{\mu',\mu} \; . \tag{14e}$$

We want to calculate the S-matrix

$$S(\mathbf{l},\sigma\,|\,\mathbf{k},s) = <\mathbf{l},\sigma\,(\mathrm{out})\,|\,\mathbf{k},s\,(\mathrm{in})> \quad . \tag{15}$$

Here $|\,\mathbf{k},s(\mathrm{in})>$ and $|\,\mathbf{k},s(\mathrm{out})>$ are solutions of the time-dependent Pauli equation with Hamiltonian (9), which approximate sharply peaked wave packets incoming and outgoing with velocity **k** at $t = \mp\,\infty$. The helicity of the incoming (outgoing) particles is fixed to be $s = \pm\,1/2$ when they are far away from the scattering centre. The crucial point is to observe that these 'distorted plane-wave-like' scattering states can be taken to satisfy the relations

$$H\,|\,\mathbf{k},s\,\binom{\mathrm{in}}{\mathrm{out}}> \quad = E_k\,|\,\mathbf{k},s\,\binom{\mathrm{in}}{\mathrm{out}}> \quad , \quad (\,k = |\mathbf{k}|\,) \tag{16a}$$

$$(\mathbf{J}_1.\mathbf{k}^\wedge)\,|\,\mathbf{k},s\,\binom{\mathrm{in}}{\mathrm{out}}> \quad = \pm\,q\,|\,\mathbf{k},s\,\binom{\mathrm{in}}{\mathrm{out}}> \quad , \quad (\mathbf{k}^\wedge = \mathbf{k}/k\,) \tag{16b}$$

$$(\mathbf{K}.\mathbf{k}^\wedge)\,|\,\mathbf{k},s\,\binom{\mathrm{in}}{\mathrm{out}}> \quad = (i\pm 1/k)\,|\,\mathbf{k},s\,\binom{\mathrm{in}}{\mathrm{out}}> \quad , \tag{16c}$$

$$(\mathbf{J}_2.\mathbf{k}^\wedge)\,|\,\mathbf{k},s\,\binom{\mathrm{in}}{\mathrm{out}}> \quad = s\,|\,\mathbf{k},s\,\binom{\mathrm{in}}{\mathrm{out}}> \quad , \tag{16d}$$

$$<\mathbf{l},\sigma\,\binom{\mathrm{in}}{\mathrm{out}}\,|\,\mathbf{k},s\,\binom{\mathrm{in}}{\mathrm{out}}> = \quad \delta_{\sigma,s}\,\delta\,(\mathbf{l}-\mathbf{k})\;. \tag{16e}$$

These relations can be made plausible by considering the formula of the symmetry generators and the meaning of the asymptotic states in a manner similar to Zwanziger's argument[2]. Alternatively, one is, in principle, able to check these formulae explicitly on concrete scattering wave functions. (In the Coulomb case these are the so-called Temple-Gordon solutions[11]).

In anyway, we *postulate* eqns. (16) and proceed to calculate the S-matrix. The method is to first expand $|\,\mathbf{k},s(\mathrm{in})>$ (resp. $|\,\mathbf{k},s(\mathrm{out})>$) in the spherical-wave basis and to calculate the matrix element (15) by using these expansions. This is a practical method since the spherical waves are easy to handle. The main convenience is that the expansions one needs can be derived by a purely algebraic argument, since all the bases which appear are labeled by suitable components of the symmetry generators which close to well-known Lie algebra.

In detail, let us first choose, for an arbitrary direction **k**, a rotation $R_\mathbf{k}$

$= \exp\{i\alpha^a(\mathbf{k})T^a\}$ which brings the **z**-direction into the direction of **k**. (The T^a, $a=1,2,3$ are 3x3 so(3) generators). The scattering centre is spherically symmetric, so one can chose scattering states satisfying the (phase convention) relations

$$|\,\mathbf{k},s\,\binom{\mathrm{in}}{\mathrm{out}}> \;=\; \exp\{\,i\,\alpha^a(\mathbf{k})\,J^a\}\,|\,k\mathbf{z}^\wedge,s\,\binom{\mathrm{in}}{\mathrm{out}}> \;. \tag{17}$$

Here the rotation matrices T^a in $R_\mathbf{k}$ are replaced by the conserved angular momentum components J^a, which generate the rotations in the quantum mechanical state space. As a consequence of eqns. (16b) and (16d), $|\,k\mathbf{z}^\wedge,s\,(\mathrm{in})>$ (resp. $|\,k\mathbf{z}^\wedge,s\,(\mathrm{out})>$) can expanded as

$$|\,k\mathbf{z},s\,\binom{\mathrm{in}}{\mathrm{out}}> \;=\; \sum_{j_1=|q|} (2j_1+1)^{1/2}\; b^{\mathrm{in,\,out}}_{j_1,\,q}\,|\,k,j_1,m_1=\pm q,\,m_2=s> \;, \tag{18}$$

and all we have to do is to determine the expansion coefficients. (The factors $(2j_1+1)^{1/2}$ are included for convenience).

Eqn. (16c) tells us how $K^3 = \mathbf{K}.\mathbf{z}^\wedge$ acts on the l.h.s. of eqn. (18). At the same time, its action on the r.h.s. of eqn. (18) is fixed by representation theory. Putting these two pieces of information together one gets recursion relations for the coefficients b. These are actually the *same* as in the spin 0 case, since the Casimirs of 0(3,1) are identical. We can take therefore, independently of s, the b 's to be those coefficients calculated by Zwanziger[2] for the spinless case,

$$b^{\mathrm{in,\,out}}_{j_1,\,s} \;=\; a^{\mathrm{in,\,out}}_{j=j_1} \;=\; \left[\frac{(j_1 \mp i/k)\,!}{(j_1 \pm i/k)\,!}\right]^{1/2} \;. \tag{19}$$

The relations $\mathbf{J} = \mathbf{J}_1+\mathbf{J}_2$, $[\mathbf{J}_1,\mathbf{J}_2] = 0$ imply that the rotation operator $\exp\{i\alpha^aJ^a\}$ factorizes as $\exp\{i\alpha^aJ^a\} = \exp\{i\alpha^aJ_1{}^a\}.\exp\{i\alpha^aJ_2{}^a\}$. This gives then rise to the equation

$$\exp\{\,i\,\alpha^a(\mathbf{k})\,J^a\}\,|\,k,j_1,m,\mu>$$

$$= \sum_{m'=-j_1}^{j_1}\;\sum_{\mu'=\pm 1/2} D^{\,j_1}_{(m',\,m)}(R_\mathbf{k})\,D^{\,1/2}_{(\mu',\mu)}(R_\mathbf{k})\,|\,k,j_1,m,\mu> \;, \tag{20}$$

with the rotation matrices D defined by $\exp\{i\alpha^a(\mathbf{k})J_1{}^a\}$ and $\exp\{i\alpha^a(\mathbf{k})J_2{}^a\}$ in the representations belonging to spin j_1 and spin $1/2$ respectively.

Collecting our results we obtain the S-matrix

$$S(\mathbf{l},\sigma \mid \mathbf{k},s) =$$

$$\delta(E_l - E_k) D_{\sigma,s}{}^{1/2}(R^{-1}{}_l R_{\mathbf{k}}) \otimes \sum_{j \geq |q|} (2j_1+1) \left[\frac{(j-i/k)!}{(j+i/k)!}\right] D^{j_1}_{(-q,q)}(R^{-1}{}_l\, R_{\mathbf{k}}), \qquad (21)$$

showing that the helicity and momentum dependence indeed factorizes. This equation can be re-written in fact as

$$S(\mathbf{l},\sigma \mid \mathbf{k},s) = D_{\sigma,s}{}^{1/2}(R^{-1}{}_l R_{\mathbf{k}}) \,.\, S_0(\mathbf{l},\sigma \mid \mathbf{k},s). \qquad (22)$$

We see that S and S_0 have the same poles, yielding the positive-energy spectrum (4).

Eqn. (22) implies that for a particle with incoming helicity s and outgoing helicity (s') the scattering cross-section is

$$(d\sigma/d\omega)_{s',s} = d\sigma_0/d\omega\, |d^{1/2}{}_{s',s}(\theta)|^2 \qquad (23)$$

where $d^{1/2}{}_{s',s}(\theta)$ is the matrix of a rotation by angle θ around the y-axis in representation spin 1/2. Since $s,s' = \pm 1/2$, $d^{1/2}{}_{1/2,1/2}(\theta) = d^{1/2}{}_{-1/2,-1/2}(\theta) = \cos\theta/2$, $d^{1/2}{}_{-1/2,1/2}(\theta) = d^{1/2}{}_{1/2,-1/2}(\theta) = \sin\theta/2$, the *total* cross-section

$$(d\sigma/d\omega)_{total} = (1/2)\sum (d\sigma/d\omega)_{s',s} = d\sigma_0/d\omega, \qquad (24)$$

identical to that in eqn. (8) for spin 0.

The zero-energy bound states do *not* appear in the S-matrix (22). This is not a surprise, since our calculations are based on the $o(3,1)\oplus o(3)$ dynamical symmetry which degenerates to $E(3)\oplus o(3)$ for ground states. So another, independent, calculation would be needed in this case.

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References

[1] Fehér L Gy 1987 Journ. Math. Phys. **28**, 234

[2] Zwanziger D 1968 Phys. Rev. **176**, 1480;
Alhassid Y, Gürsey F and Iachello F 1983 Ann. Phys. (N.Y.) **148**, 356
Mladenov I 1987 Sofia Preprint

[3] McIntosh H V and Cisneros A 1970 Jour. Math. Phys. **11**, 890
Barut A O and Bornzin G L 1971 Journ. Math. Phys. **12**, 841
Schönfeld J F 1980 Journ. Math. Phys. **21** 2528;
Bates L 1987 Calgary preprint

[4] Fehér L Gy 1986 J. Phys. **A19**, 1259 ;
1987 in Proc. Siófok Conference on Non-Perturbative methods in Quantum Field Theory, ed. Z. Horváth, L. Palla and A. Patkós, World Scientific

[5] Pauli W 1926 Z. Phys. **36**, 33;
Bargmann V 1936 Z. Phys. **99**, 570

[6] Gibbons G W and Manton N S 1986 Nucl. Phys. **B274**,183;
Fehér L Gy and Horváthy P A 1987 Phys. Lett. **182B**, 183;
(E) **188B**, 512; Gibbons G W and Ruback P 1987 Phys. Lett. **188B**, 226; Cordani B, Fehér L Gy and Horváthy P A 1987 Phys. Lett. **B** (submitted)

[7] D'Hoker E and Vinet L 1985 Phys. Rev Lett. **55**, 1043;
1985 Nucl. Phys. **B260**, 79

[8] Fehér L Gy and Horvathy P A (to be published)

[9] D'Hoker E and Vinet L 1986 Lett. Math. Phys. **12**, 71

[10] Naimark M A 1957 *Linear representations of the Lorentz Group* , Amer. Math. Soc., Providence, R.I., Chapter 3. Sec. n°3.

[11] Mott N F and Massey H S W 1965 *The theory of atomic colision* 3rd ed., Ch. XI, Oxford University Press